Instanton interactions and Borel summability

Zbigniew Ambroziński

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INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

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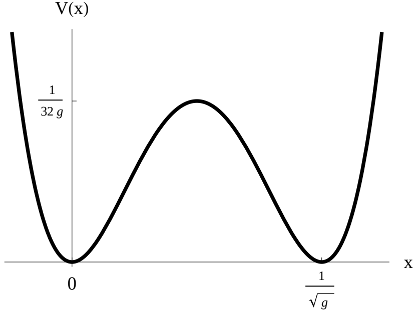
Plan of the seminar

Anharmonic double well potential

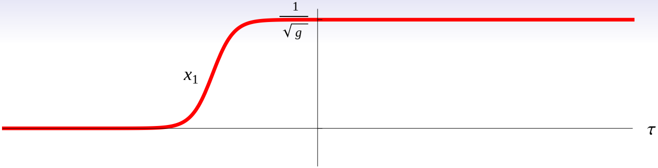
$$V(x) = \frac{1}{2}x^2(1-\sqrt{g}x)^2$$



Energies to be determined from integral in Euclidean space.

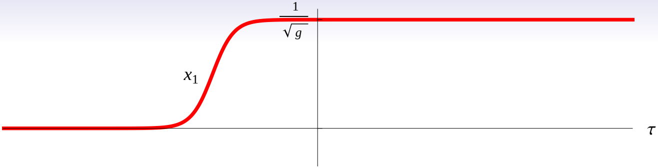
$$c_0 e^{-E_0 T} + c_1 e^{-E_1 T} = \int \mathcal{D}[x(\tau)] e^{-S[x(\tau)]}$$

$$S[x(\tau)] = \int_{-T/2}^{T/2} d\tau \left(\frac{1}{2}\dot{x}^2(\tau) + V(x(\tau))\right)$$



Action of an instanton

$$S[x_1] = S_0 = 1/6g$$



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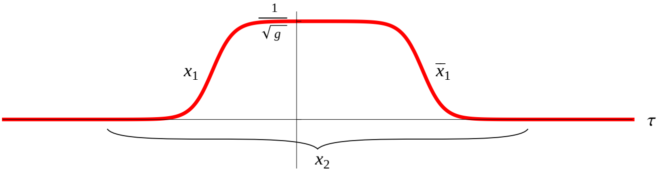

Action of n instantons

$$S[x_n] = nS_0 = n/6g$$

$$c_0 e^{-E_0 T} + c_1 e^{-E_1 T} = \int \mathcal{D}[x(\tau)] e^{-S[x(\tau)]}$$

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$$= \underbrace{\sum_n}_{} \underbrace{\int_{-T/2}^{T/2} dt_1 \underset{t_i < t_{i+1}}{\ldots} dt_{2n}}_{} \underbrace{e^{-2nS_0}}_{} \underbrace{c\kappa^{2n}}_{} e^{-T/2}$$

- $\sum_n$: sum over classical solutions
- $\int dt_1 \ldots dt_{2n}$: integration over zero modes
- e^{-2nS_0}: classical action
- $c\kappa^{2n}$: van Vleck determinant

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$$= ce^{-T/2}\cosh(\kappa T e^{-S_0})$$

Energies shifted by nonperturbative quantity compared to $E(g=0)$.

$$E_0 = \frac{1}{2} - \frac{1}{\sqrt{\pi g}} e^{-1/6g}$$

$$E_1 = \frac{1}{2} + \frac{1}{\sqrt{\pi g}} e^{-1/6g}$$

True energies [Zinn-Justin]

$$E_0 = \sum_n a_n g^n - \frac{1}{\sqrt{\pi g}} e^{-1/6g} \sum_n b_n g^n = E_{pert.} - \delta_1 E$$

$$E_1 = \sum_n a_n g^n + \frac{1}{\sqrt{\pi g}} e^{-1/6g} \sum_n b_n g^n = E_{pert.} + \delta_1 E$$

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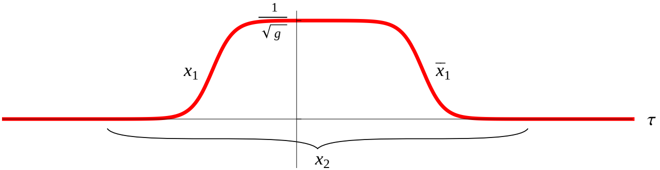
Instanton calculus gives complete information about difference of energies for small g:

$$\Delta E = E_1 - E_0 = \frac{2}{\sqrt{\pi g}} e^{-1/6g} \sum_n b_n g^n$$

Interaction of instantons



Action of two instantons

$$S[x_2] = 2/6g - 2e^{-|t_1 - t_2|}/g \qquad \text{[Bogomolny]}$$

Contribution to the integral coming from two instantons

$$Z_2 = \int_{-T/2}^{T/2} dt_1 dt_2 e^{-S[x_2]} \approx \frac{1}{2}(Te^{-S_0})^2 + Te^{-2S_0} I(g)$$

$$I(g) = \int_0^\infty dR(\exp(2e^{-R}/g) - 1)$$

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Dilute gas approximation:

$g > 0$	$\Rightarrow$	main contribution from small R	WRONG
$g < 0$	$\Rightarrow$	main contribution from large R	OK

Calculate the integral for negative g and continue to positive g

$$I(g) \approx -\gamma + \ln(-g/2) = -\gamma + \ln(g/2) \pm i\pi$$

$$E_0 = \frac{1}{2} - \frac{1}{\sqrt{g\pi}} e^{-1/6g} + \frac{1}{g\pi} e^{-1/3g}(\gamma + \ln(-2/g))$$

$$E_1 = \frac{1}{2} + \frac{1}{\sqrt{g\pi}} e^{-1/6g} + \frac{1}{g\pi} e^{-1/3g}(\gamma + \ln(-2/g))$$

$$
\begin{aligned}
E_{0,1} &= \sum_n a_n g^n \pm \frac{1}{\sqrt{g\pi}} e^{-1/6g} \sum_n b_n g^n \\
&\qquad + \frac{1}{g\pi} e^{-1/3g} (\gamma + \ln(-2/g)) \sum_n c_n g^n \\
&= E_{pert.} \pm \delta_1 E + \delta_2 E \qquad \text{[Zinn-Justin]}
\end{aligned}
$$

Why should we bother with instanton interactions?

- Much smaller than perturbative energy and free instantons.
- No chance to see them numerically !?

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But

- Borel series ambiguity cured. [Ünsal]
- Imaginary part cancels.
- Real part actually can be seen in energy average!

Perturbative expansion

$$H\psi = E\psi$$

$$\left(\frac{1}{2}P^2 + V(X)\right)\sum_{n,k}\alpha_{n,k}g^n\,|k\rangle = \sum_n \epsilon_n g^n \sum_{n,k}\alpha_{n,k}g^n\,|k\rangle$$

Solve this equation recursively to obtain $\alpha_{n,k}$ and ϵ_n.

$$E = \sum_n \epsilon_n g^n$$

But: this is nonalternating(!) asymptotic series. [Bender, Wu]

ϵ_0	$1/2$
ϵ_1	-1
ϵ_2	$-9/2$
ϵ_3	$-89/2$
ϵ_4	$-5013/8$
ϵ_{350}	$-1.15 \times 10^{907} \approx -0.95 \times 3^{350} \times 350!$

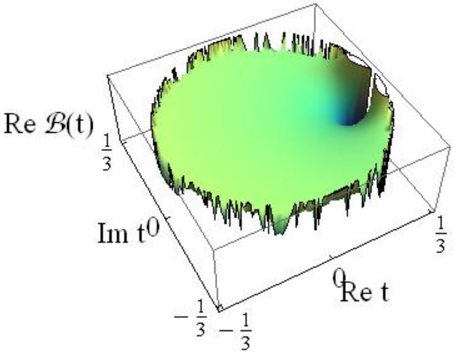
Borel transform

$$\mathcal{B}(t) = \sum_n \frac{1}{n!}\epsilon_n t^n$$

Inverse Borel transform needs $\mathcal{B}(t)$ on the positive real axis

$$E_{Borel}(g) = \frac{1}{g}\int_0^\infty dt e^{-t/g}\mathcal{B}(t)$$

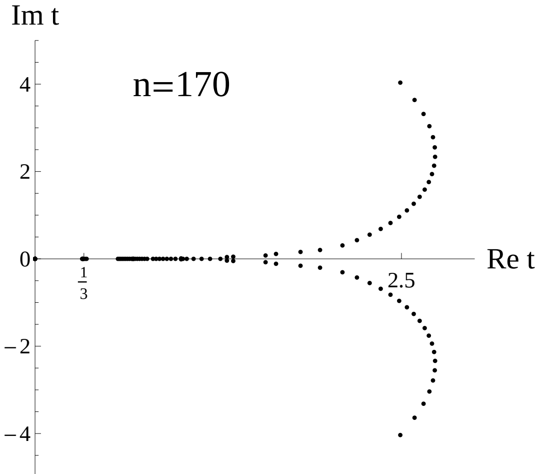
But:
$\mathcal{B}(t)$ has radius of convergence only $1/3$.



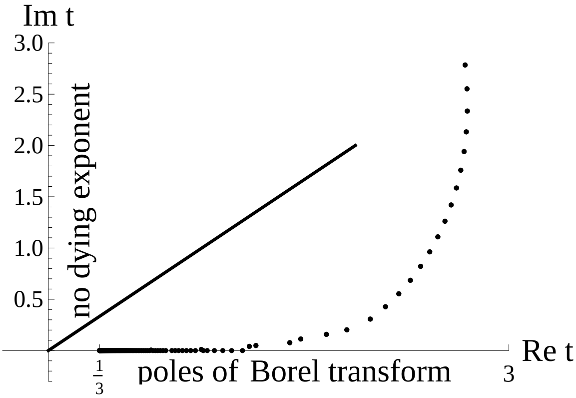
Pade approximant

Padé approximant provides information about analytical continuation of Borel transform.

$\mathcal{P}_n(t)$ ratio of polynomials of order n, $\mathcal{P}^{(i)}(0) = \mathcal{B}^{(i)}(0)$: $i \leqslant 2n$



Analytical continuation of Borel transform has a cut at $(1/3, \infty)$.
$\Rightarrow$ Integrate along a contour.



Integration limit $Re\ t < 2 \Rightarrow$ error of order $e^{-2/g}$ – unimportant.

Borel sum and instantons

Borel sum

E_{Borel} can be calculated only for $t = t + i\epsilon$ or $t = t - i\epsilon$.

If $\mathcal{B}(t)$ had only a simple pole at $t = 1/3$ the two cases would differ by $\frac{i}{g} e^{-1/3g} \times const.$

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Instanton interaction

Instanton interactions exist in limits $g = g + i\epsilon$ and $g = g - i\epsilon$!

$\delta_2 E$ in these cases differ by $\frac{2i}{g} e^{-1/3g}$

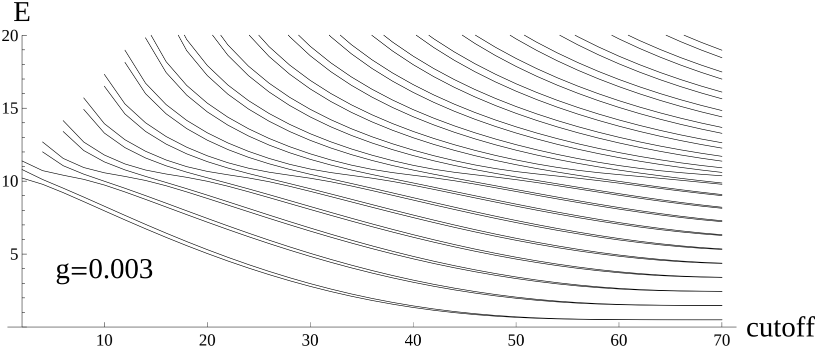
Do these two ambiguities cancel?

Cut Fock space method

- express Hamiltonian as a matrix:

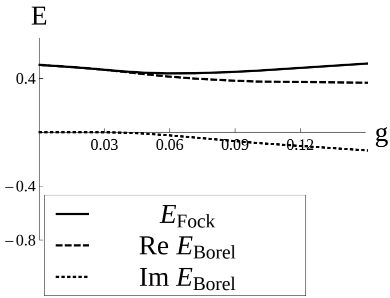
$$(H)_{m,n} = \langle m | \frac{1}{2} P^2 + V(X) | n \rangle$$

- introduce cutoff to get finite matrix
- eigenvalues approximate energies [Wosiek]

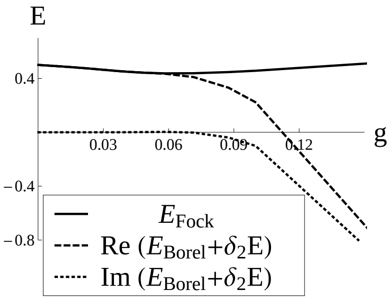


Is instanton contribution needed?

without $\delta_2 E$



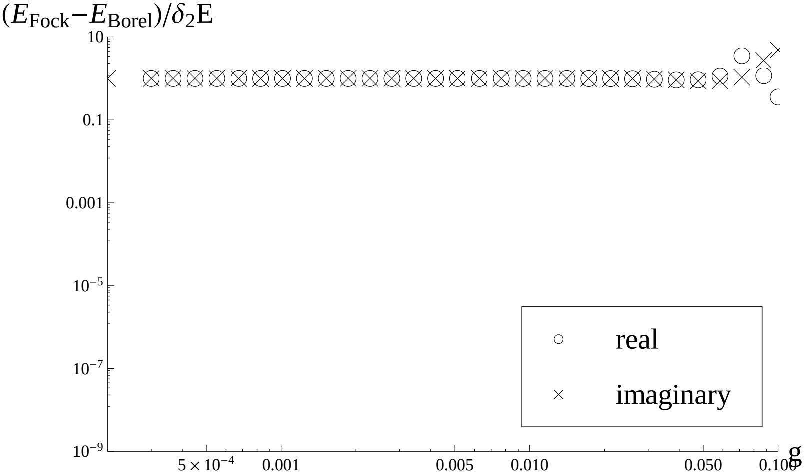
with $\delta_2 E$



Adding $\delta_2 E$ improves result for $g < 0.07$ and worsens for $g > 0.09$.

High precision comparison

without $\delta_2 E$



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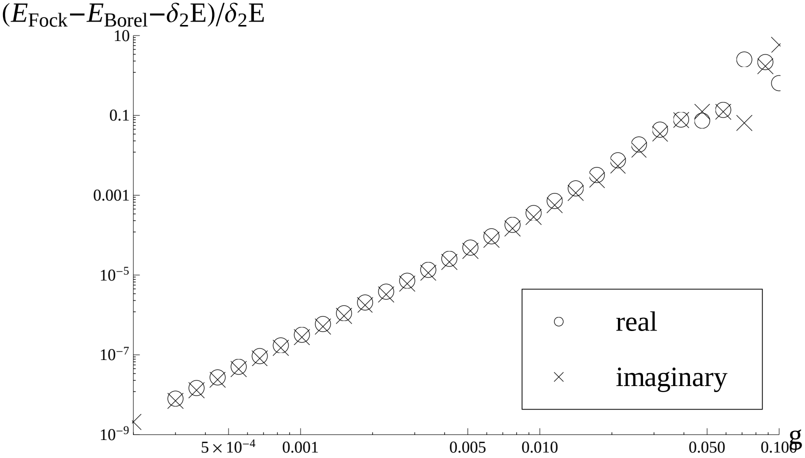

Next correction is seen!

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- instanton interactions cancel ambiguity of Borel energies
- instanton interactions improve Borel energies for small g
- for larger g the (asymptotic) series $\sum c_n g^n$ becomes important and needs to be summed (Borel sum?)
- but only few c_n are known
- next corrections: 3–instanton interactions

Literature

- E.B. Bogomolny *Calculation if instanton - anti–instanton contributions in quantum mechanics*, Phys.Lett. B91 (1980) 431-435
- J. Zinn-Justin, *Multi - instanton contributions in quantum mechanics*, Nucl.Phys. B192 (1981) 125-140;
  J. Zinn-Justin, U.D. Jentschura, *Higher order corrections to instantons*, J.Phys.A A34 (2001) L253-L258
- M. Ünsal, *Theta dependence, sign problems and topological interference*, arXiv:1201.6426
- C. M. Bender, T. T. Wu, *Anharmonic oscillator*, Phys.Rev. 184 (1969) 1231-1260
- M. Trzetrzelewski, J. Wosiek, *Quantum systems in a cut Fock space*, Acta Phys.Polon. B35 (2004) 1615-1624;
  J. Wosiek, *Spectra of supersymmetric Yang-Mills quantum mechanics*, Nucl.Phys. B644 (2002) 85-112